THEORY OF A LAVAL NOZZLE FOR A TWO-PHASE MIXTURE CONTAINING PARTICLES OF SMALL LAG

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A two-velocity and two-temperature model is considered for a continuous medium in relation to the flow of a mixture of gas and particles in the subsonic, transsonic, and supersonic parts of a Laval nozzle. It is assumed that the particles are small, and hence that the coefficients φ^f and φ^q, which define the interaction with the gas, are large (these coefficients are inversely proportional to the square of the particle radius for a Stokes mode of flow). This means that the velocity or thermal lag of the particles relative to the gas is small. The solution is sought as expansions with respect to the small parameters $\varepsilon_1 = 1/\varphi^f$ and $\varepsilon_2 = 1/\varphi^q$.

The first problem is one of special perturbations, which arise from the formation of a layer of pure gas near the wall on account of particle lag; if $\varepsilon_1 \to 0$, the thickness of this layer tends to zero, but the difference in the values of the gas parameter at the wall and at the boundary of the layer remains finite. Equations are derived that describe with accuracies ε_1 and ε_2 the flow of the mixture of gas and particles at the core, together with equations that define the gas parameters in the wall layer. It is found that this layer resembles an ordinary boundary layer in that the pressure change across it is a quantity of higher order than the change in the other parameters. To solve the equations that define the characteristics of the mixture at the core, use is made of an expansion with respect to the small parameter $\varepsilon = 1/R$, where R is the radius of curvature of the nozzle wall in the minimal cross section, as referred to the radius at that section.

The two-phase flow in a Laval nozzle has been considered [1-3] in the one-dimensional approximation by expansion with respect to ε_1 and ε_2 for one of these parameters; the expansion with respect to ε used here is analogous to the method of [4] for solving problems in the theory of Laval nozzles for pure gas. Exact published results are available within the framework of the two-liquid model for the flow of a mixture of gas and particles, but these are restricted to the supersonic part of the nozzle, where the parameters of the gas and particles are derived by the direct methods of characteristics [5-11] or by inverse methods [12-14].

On the other hand, we are not aware of any papers in which solutions have been obtained for the theory of Laval nozzles for nonequilibrium two-phase flow for the supersonic and subsonic or transsonic parts without involving additional assumptions; of the available approximate approaches, we may note the very widely used method of integrating the equations of energy and motion for the particles (these equations are ones in total derivatives along the particle flow lines), which are used for the equilibrium parameters of the mixture [15], while there is also the method of [16], in which it was assumed that the distributions of the pressure and the inclination of the gas velocity vector are the same for the equilibrium and nonequilibrium two-phase flows. There is some justification for using these approximate approaches for comparatively small relative flow rates of the particles, but at high relative flow rates the assumptions may involve substantial errors.

The presence of a layer of pure gas near the wall also hinders the formulation and solution of the inverse problem, while the method of [17-19], which is very effective for a pure gas, becomes difficult to use when there is little particle lag and the layer is thin. It may be that the relationships derived below

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for the wall layer should be used in conjunction with the methods noted above. In this connection we must stress that these relationships apply for any ε_1 when the layer is thin by comparison with the characteristic dimension of the nozzle. This arises because ε_1 appears in the equations only via the thickness of the layer when there are no particles, and this thickness, as we shall see, is proportional to ε_1 when the particle lag is small.

1. We use a rectangular or cylindrical system of coordinates to consider the flow of gas mixed with particles in a planar or axially symmetrical Laval nozzle (Fig. 1). We locate the origin in the plane of minimum cross section of the nozzle, with the x axis directed from left to right (along the flow) along the axis or symmetry plane, while the y axis is perpendicular to the x axis. If there are no external sources of heat or force, and if we neglect the volume of the particles, we get the following equations for the flow within the framework of the two-fluid model [5-14, 20]:

$$
\begin{gathered}
u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\rho_s}{\rho}\varphi^f(u-u_s)=0\\
u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\rho_s}{\rho}\varphi^f(v-v_s)=0\\
\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\nu\frac{\rho v}{y}=0\\
u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}-\frac{u}{\rho}\frac{\partial p}{\partial x}-\frac{v}{\rho}\frac{\partial p}{\partial y}-\frac{\rho_s}{\rho}[\varphi^f(u-u_s)^2+\varphi^f(v-v_s)^2+\varphi^q(T_s-T)]=0\\
u_s\frac{\partial u_s}{\partial x}+v_s\frac{\partial u_s}{\partial y}=\varphi^f(u-u_s),\qquad u_s\frac{\partial v_s}{\partial x}+v_s\frac{\partial v_s}{\partial y}=\varphi^f(v-v_s)\\
\frac{\partial \rho_s u_s}{\partial x}+\frac{\partial \rho_s v_s}{\partial y}+\nu\frac{\rho_s v_s}{y}=0,\quad u_s\frac{\partial e_s}{\partial x}+v_s\frac{\partial e_s}{\partial y}=\varphi^q(T-T_s)\\
p=\rho T,\quad h=\frac{\varkappa}{\varkappa-1}T,\quad e_s=\delta T_s
\end{gathered}
\tag{1.1}
$$

Here p is pressure, h is specific enthalpy, T is temperature, ρ is density, u and v are the projections of the gas velocity vector on the x and y axes, T_s, ρ_s, u_s, and v_s are the corresponding quantities for the particles, e_s is the specific internal energy of the particles, and the coefficients φ^f and φ^q, which we take to be constants, which corresponds to Stokes flow around each particle, characterize the dynamic and thermal interaction between the gas and the particles; $\nu = 0$ or 1 respectively in the planar and axially symmetrical cases. We assume that the gas is perfect, with constant specific heats and ratio $\varkappa$, while the internal energy of the particles is a linear function of the temperature (δ is a constant equal to the specific heat of the particles).

All the quantities are taken as dimensionless in (1.1) and subsequently. Let L, q_*, and ρ_* be characteristic quantities with the dimensions of length, velocity, and density, while R is the dimensional value of the gas constant. Then we perform the reduction to dimensionless form by referring the spatial variables to L, the velocities to q_*, the densities to ρ_*, the pressures to $\rho_* q_*^2$, the enthalpy and internal energy to q_*^2, the temperature to q_*^2/R, and the specific heat of the particles to R. As L we take the radius or half height of the minimal section of the nozzle, while as q_* and ρ_* we take the critical velocity and density for the equilibrium flow, i.e., flow without particle lag, where $u_s \equiv u$, $v_s \equiv v$, and $T_s \equiv T$.

The boundary conditions for (1.1) are the condition for absence of flow at the wall, which is specified by the equation $y = y_w(x)$, and the symmetry condition for $y = 0$, these taking the form

$$
v(x, y_w) = y_w'(x)\,u(x, y_w),\quad v(x, 0) = 0 \tag{1.2}
$$

and

$$
u_s = u,\quad v_s = v,\quad T_s = T\quad (x \to -\infty) \tag{1.3}
$$

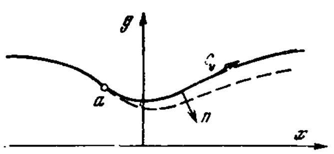

Fig. 1

The prime in (1.2) and subsequently denotes the total derivative with respect to the corresponding argument (in this case x).

Equation (1.3) applies when the nozzle is joined on the left to a semi-infinite cylindrical tube (here the vertical components of the velocities of gas and particles tend to zero, while the limiting values of the horizontal components differ from zero) and when the nozzle expands without limit for $x \to -\infty$ (then both components of the velocities tend to zero). Apart from (1.2) and (1.3), we have to specify the entropy and total enthalpy, and also the density ratio of the particles and gas (ρ_s/ρ) at the inlet to the nozzle (for $x \to -\infty$). We restrict ourselves to the case where these quantities are constant over the cross section (a uniform flow at the inlet) and take account of the choice of q_* and ρ_* to write these conditions in the form

$$\frac{\varkappa_e p}{(\varkappa_e - 1)\rho_\Sigma} + \frac{q^2}{2} = \frac{\varkappa_e + 1}{2(\varkappa_e - 1)}, \qquad \frac{p}{(\rho_\Sigma)^{\varkappa_e}} = \frac{1}{\varkappa_e}$$
$$\frac{\rho_s}{\rho} = \frac{1-m}{m} \qquad (x \to -\infty) \tag{1.4}$$
$$(\rho_\Sigma = \rho + \rho_s, \quad q = (u^2 + v^2)^{1/2})$$

Here ρ_Σ is the total density of the mixture, while q is the modulus of the velocity (q tends to u or to 0 for $x \to -\infty$), m is a specified constant that does not exceed unity (gas flow rate), and $\varkappa_e$ is the adiabatic parameter of the equilibrium two-phase mixture:

$$\varkappa_e = \frac{\alpha}{\alpha - 1} \qquad \left(\alpha = \frac{\varkappa}{\varkappa - 1} + \frac{1-m}{m}\delta\right)$$

2. If φ^f is finite, a layer free from particles is formed near convex parts of the wall on account of particle lag, this layer being bounded at the top by the wall of the nozzle, at which condition (1.2) is obeyed for the gas. Downwards, this layer of pure gas is bounded by the line $y = y_d(x)$, which is the limiting current line for the particles, which is shown in Fig. 1 as a broken line defined by the equation

$$y_d' = (v_s / u_s)_d \tag{2.1}$$

Here the subscript d denotes the parameters on the separation line. In the general case, the properties of system (1.1) mean that ρ_s changes abruptly on passing through this boundary from some finite value to zero, which leads to discontinuity in the first derivatives for the gas parameters. We shall see below that the gas enters the wall layer and thickens it when the particle lag is small near the convex part [$y_w''(x) \geq 0$].

To describe the flow in the region free from particles we use a coordinate system linked to the contour of the nozzles τn, where τ lies along the wall (from left to right, as shown in Fig. 1), while n lies along the normal to the latter into the gas. If σ is the angle between the internal normal to the contour (i.e., $-\mathbf{n}$) and the y axis, then the relation between x, y, τ, and n is

$$x = x_w(\tau) + n \sin\sigma, \quad y = y_w(\tau) - n\cos\sigma$$
$$\frac{dx_w}{d\tau} = \cos\sigma = (1 + y_w'^2)^{-1/2}, \qquad \frac{dy_w}{d\tau} = \sin\sigma = y_w' \cos\sigma \tag{2.2}$$

The following formulas relate the projections of u, v and U, V of the gas velocity vector in the xy and τn systems respectively:

$$U = u\cos\sigma + v\sin\sigma, \quad V = u\sin\sigma - v\cos\sigma$$
$$u = U\cos\sigma - V\sin\sigma, \quad v = U\sin\sigma - V\cos\sigma$$

Then these relationships give the following form to the equations describing the gas flow in the wall layer where $\rho_s \equiv 0$:

$$U\frac{\partial U}{\partial \tau} + V(1 + nK)\frac{\partial U}{\partial n} + UVK + \frac{1}{\rho}\frac{\partial p}{\partial \tau} = 0$$
$$U\frac{\partial V}{\partial \tau} + V(1 + nK)\frac{\partial V}{\partial n} - U^2K + \frac{1 + nK}{\rho}\frac{\partial p}{\partial n} = 0$$
$$U\frac{\partial h}{\partial \tau} + V(1 + nK)\frac{\partial h}{\partial n} - \frac{U}{\rho}\frac{\partial p}{\partial \tau} - \frac{V(1 + nK)}{\rho}\frac{\partial p}{\partial n} = 0 \tag{2.3}$$
$$\frac{\partial}{\partial \tau}(y^\nu \rho U) + \frac{\partial}{\partial n}[y^\nu(1 + nK)\rho V] = 0 \qquad \left(K = \frac{y_w''}{(1 + y_w'^2)^{3/2}}\right)$$

Here K is the curvature of the nozzle wall.

From the first three equations of (2.3) we obtain in the usual way the condition for confirmation of the total enthalpy, while the third equation is used with the definition of entropy and the absence of irreversible processes in the wall layer to derive the entropy conservation condition along each current line:

$$2h + U^2 + V^2 = 2\,H(\psi), \quad s = S(\psi) \tag{2.4}$$

Here s is the specific entropy of the gas, which is a known function of p and ρ, while the current function ψ is defined by the following differential equation in accordance with the last equation in (2.3):

$$d\psi = -\,cy^\nu \rho U dn + cy^\nu (1 + nK)\,\rho V d\tau \tag{2.5}$$

and by specification of the current function ψ_W at the wall; the functions $H(\psi)$ and $S(\psi)$ are determined by the total enthalpy and entropy at the point of intersection of the given current line with the interface. The constant c in (2.5) is best taken such that ψ_W is unity when ψ equals 0 at the symmetry axis.

In the new variables, the condition for absence of flow at the nozzle wall, i.e., the first equation from (1.2), is written in the form

$$V_w \equiv V(\tau, 0) = 0 \tag{2.6}$$

3. The solution to (1.1) that describes the flow of the two-phase mixture outside the wall layer of pure gas will be derived by expanding all the dependent variables with respect to the parameters $\varepsilon_1 = 1/\varphi^f$ and $\varepsilon_2 = 1/\varphi^q$; the corresponding expansions are taken in the form

$$\begin{gathered}\omega = \omega_e + \Delta\omega + \ldots, \quad \rho = \rho_e(1 + \Delta\rho + \ldots) \\ \rho_s = \rho_{se}(1 - \Delta\rho_s + \ldots)\end{gathered} \tag{3.1}$$

where ω is any parameter other than ρ and ρ_S; $\Delta\varphi = \varepsilon_1\varphi_1 + \varepsilon_2\varphi_2$, and the sets of three dots denote terms of higher order of smallness in ε_1 and ε_2.

The terms with the subscript e in (3.1) correspond to equilibrium flow, which occurs when $\varepsilon_1 \equiv \varepsilon_2 \equiv 0$, with no lag of the particles in velocity or temperature. These parameters are determined [2a] by the flow equations for a perfect gas with $\varkappa_e$ for the ratio of the specific heats, density $\rho_{\Sigma e}$, and enthalpy

$$h_{\Sigma e} = \varkappa_e p_e / (\varkappa_e - 1)\,\rho_{\Sigma e}$$

When there is uniform flow at $x = -\infty$, i.e., when (1.4) is obeyed, the calculation for an equilibrium two-phase flow is similar to that for a pure gas in amounting to integration of two differential equations (for continuity and absence of vortices), which can be put [1] in the form

$$\begin{gathered}\left(1 - u_e^2 - \frac{\varkappa_e - 1}{\varkappa_e + 1} v_e^2\right)\frac{\partial u_e}{\partial x} + \left(1 - v_e^2 - \frac{\varkappa_e - 1}{\varkappa_e + 1} u_e^2\right)\frac{\partial v_e}{\partial y} + \left[1 - \frac{\varkappa_e - 1}{\varkappa_e + 1}(u_e^2 + v_e^2)\right]\frac{\nu v_e}{y} - \frac{4u_e v_e}{\varkappa_e + 1}\frac{\partial u_e}{\partial y} = 0 \\ \frac{\partial u_e}{\partial y} - \frac{\partial v_e}{\partial x} = 0\end{gathered} \tag{3.2}$$

Here the other parameters are expressed in terms of u_e and v_e by the finite relations

$$\begin{gathered}\rho_{\Sigma e} = \left(\frac{\varkappa_e + 1}{2} - \frac{\varkappa_e - 1}{2} q_e^2\right)^{1/(\varkappa_e - 1)}, \quad q_e^2 = u_e^2 + v_e^2, \quad \rho_e = m\rho_{\Sigma e} \\ \rho_{se} = (1 - m)\,\rho_{\Sigma e}, \quad p_e = \frac{(\rho_{\Sigma e})^{\varkappa_e}}{\varkappa_e}, \quad T_e = \frac{p_e}{\rho_e} \\ h_e = \frac{\varkappa_e}{\varkappa_e - 1} T_e, \quad u_{se} = u_e, \quad v_{se} = v_e, \quad T_{se} = T_e\end{gathered} \tag{3.3}$$

The equations for the next terms in the expansions for ρ_1, p_1, ... and ρ_2, p_2... are found by substituting (3.1) into (1.1) and equating the terms to the first powers of ε_1 and ε_2 respectively. The resulting linear equations for the coefficients to ε_1 and ε_2 may be combined as equations for $\Delta\rho$, Δp, ..., and these can be transformed to a system of five linear differential equations for $\Delta\rho$, Δp, Δu, Δv, and $\Delta\rho_s$, together with five finite relationships for the other parameters. The differential equations can be put in the form

$$L_i(\omega_e, \Delta\omega, \Delta\rho, \Delta\rho_s) = 0 \tag{3.4}$$

Here L_i are differential operators ($i = 1, \ldots, 5$), which are quasilinear with respect to ω_e and linear with respect to the other variables; by ω_e we understand u_e and v_e, while by ω we understand any of the parameters p, u, and v, or sets of these. The definition of $\Delta\varphi$ means that the terms in (3.4) containing only the equilibrium quantities (i.e., ω_e and their derivatives) enter into the equations with the factors ε_1 and ε_2.

The following are the finite relationships defining the deviations from equilibrium in the other parameters of the mixture:

$$\begin{aligned}
&\Delta T = \frac{\Delta p}{\rho_e} - \frac{p_e}{\rho_e}\Delta\rho, \quad \Delta h = \frac{\varkappa p_e}{(\varkappa - 1)\rho_e}\left(\frac{\Delta p}{p_e} - \Delta\rho\right) \\
&\Delta u_s = \Delta u - \varepsilon_1 u_e u_e', \quad \Delta v_s = \Delta v - \varepsilon_1 u_e v_e' \\
&\Delta T_s = \Delta T + \varepsilon_2 \frac{\varkappa_e - 1}{\varkappa_e m}\delta u_e q_e q_e^2
\end{aligned} \tag{3.5}$$

where the prime denotes a total derivative with respect to x along the current line for the equilibrium flow, the defining equation being $y' = v_e/u_e$.

Equation (2.1) defines the form of the interface, and it gives the following equation when the third and fourth relationships from (3.5) are incorporated together with the condition from (1.2) for the absence of flow at the wall, omitting small terms of higher order:

$$y_w' - y_d' = \varepsilon_1 u_{we} y_w'' \tag{3.6}$$

which shows that the thickness of the wall layer along y, i.e., the difference $y_w - y_d$, and hence also along n, is of the order of ε_1, and in this approximation increases (decreases) on parts with positive (negative) curvature of the wall. From (3.6) we find that the layer free from particles first appears at the point where $y_w'' = 0$ if to the left of that point one has y_w'' everywhere negative (point *a* in Fig. 1). If $y_w'' \geq 0$ everywhere along the contour, as in the nozzle considered below with hyperbolic generator, then the wall layer of pure gas has zero thickness only at an infinitely remote point (for $x \to -\infty$). It will become clear below that the changes transverse to the wall layer are quantities of the order $0(1)$ apart from the pressure, thermodynamic parameters of the gas, and velocity component U. The thickness of this layer is $n_d = 0(\varepsilon_1)$, so we get first of all equations that enable one to find the corresponding parameters to $0(\varepsilon_1^0) = 0(1)$; it will be shown at the end of the following section that a simple modification to the equations enables one to calculate the parameters in the wall layer with an error of order $0(\varepsilon_1, \varepsilon_2)$.

It follows from $n_d \sim \varepsilon_1$ in conjunction with the last equation of (2.3) and the zero flow condition of (2.6), as for an ordinary boundary layer, that $V \sim \varepsilon_1$ in the wall layer; then the second equation in (2.3) gives that the change in p transverse to the layer is of the order of ε_1, i.e., $p_w - p_d = 0(\varepsilon_1)$, so to an accuracy $0(1)$

$$p(\tau, n) = p_{we}(\tau) \qquad (0 \leqslant n \leqslant n_d) \tag{3.7}$$

while the other equations of (2.3) or the equivalent ones (2.4) take the form

$$2h(p_{we}, \rho) + U^2 = 2H(\psi), \quad s(p_{we}, \rho) = S(\psi) \tag{3.8}$$

The functions $H(\psi)$ and $S(\psi)$ on the right in (3.8) are calculated in this approximation from the equilibrium values for the total enthalpy and entropy of the gas at the nozzle wall (the corresponding quantities at the wall and at the interface line differ by the order of ε_1 for an equilibrium flow), i.e.,

$$2H(\psi) = (2h + U^2)_{we}, \quad S(\psi) = s_{we} \tag{3.9}$$

One can relate the functions of x on the right in (3.9) and ψ when we have determined $\psi_d = \psi_d(x)$; the latter is found in each section $\tau = \tau(x_w) = \text{const}$ by integration from $n = 0$, where $\psi(\tau, 0) = \psi_w$, to $n = n_d$ for (2.5), the result to terms in ε_1 being put in the form

$$d\psi = -cy_w^{\nu}\,\rho U dn \tag{3.10}$$

Here n_d is calculated from y_d as found from (3.6) using (2.2). One performs a sequential calculation for sections τ equals constant in order of increasing τ from the point where the wall layer appears via (3.6)-(3.10), which enables one to determine $H(\psi)$ and $S(\psi)$, and the distribution in n for all parameters apart from $V \sim \varepsilon_1$ in each cross section.

The wall layer of low-entropy gas influences the entire flow, producing parameter changes of the order of ε_1; when the calculations have been performed for this layer, the effects are considered, as for an ordinary boundary layer via the displacement thickness

$$\delta^*(\tau) = \int_0^{n_d} \left[1 - \frac{\rho U}{(\rho U)_{we}}\right] dn$$

and successive calculations for the equilibrium flow, the process being performed for a contour derived from the initial one by adding δ^*, which at each point is marked out along the normal $\mathbf{n}$. If δ^* is positive (negative), i.e., y_w, then the ordinate is reduced (increased). If this conversion is not performed, one has, as for a boundary layer, that the results correspond to a contour deformed analogously by $-\delta^*$.

It is clear from this analysis that everything said in this section about the wall layer applies for any ε_1, an arbitrary resistance law for the particles, and so on provided that the layer thickness is small (if the gas expansion is sufficiently smooth, this often applies even for comparatively large ε_1). In the corresponding arguments, one should understand by ε_1 the thickness of the wall layer, as referred to the characteristic dimension of the nozzle. On the other hand, it is obligatory for ε_1 and ε_2 to be small for the equations describing the flow at the core to apply, although the results are applicable for any resistance law and any mode of heat transfer for the particles. This arises because any of the laws give formulas corresponding to Stokes flow after appropriate linearization if there is only slight dynamic and thermal lag in the particles.

4. In the calculations whose results are given below we have taken the solution to (3.2) as calculated from formulas analogous to those of [4], in which an expansion was used with reference to the parameter $\varepsilon = 1/R$, where R is the radius of the curvature in the minimal cross section (at $x = 0$), as referred to the radius (for $\nu = 1$) or the half height (for $\nu = 0$) for the nozzle in that section. When we have introduced new variables ($z = xR^{-1/2}$ and $w = vR^{1/2}$), these formulas become up to ε as follows:

$$\begin{aligned} u_e(z, y) &= a_0(z) + \varepsilon\,[b_0(z) + y^2 b_1(z)] \\ w_e(z, y) &= y a_1(z) + \varepsilon y\,[b_2(z) + y^2 b_3(z)] \end{aligned} \tag{4.1}$$

where the symbols used here and subsequently differ from those of [4].

The function a_0 gives the distribution in z for the axial component of the velocity in the one-dimensional approximation, and this is found from the ordinary differential equation

$$(1 - a_0^2)\, a_0' + (1 + \nu)\left(1 - \frac{\varkappa_e - 1}{\varkappa_e + 1}\, a_0^2\right) a_0 (\ln y_w)' = 0 \tag{4.2}$$

which is integrated subject to the initial condition $a_0(0) = 1$; function a_1 is expressed via a_0 as $a_1 = a_0 (\ln y_w)'$

Note that here the prime denotes derivatives with respect to z.

When a_0 and a_1 have been determined, functions b_i in (4.1) are found from a solution of one differential equation and three finite equations:

$$\begin{aligned} &(1 - a_0^2)\, b_0' = 2a_1 b_0 \left[a_0' + (1 + \nu)\frac{\varkappa_e - 1}{\varkappa_e + 1}\, a_1\right] - (1 + \nu)\left(1 - \frac{\varkappa_e - 1}{\varkappa_e + 1}\, a_0^2\right) b_2 \\ &b_1 = a_1'/2 \\ &(3 + \nu)\left(1 - \frac{\varkappa_e - 1}{\varkappa_e + 1}\, a_0^2\right) b_3 = \frac{8}{\varkappa_e + 1}\, a_0 a_1 b_1 - (1 - a_0^2)\, b_1' + \\ &+ 2a_0 b_1 \left[a_0' + \frac{\varkappa_e - 1}{\varkappa_e + 1}(1 + \nu)\, a_1\right] + a_1^2 \left[\frac{\varkappa_e - 1}{\varkappa_e + 1}\, a_0' + \left(1 + \frac{\varkappa_e - 1}{\varkappa_e + 1}\, \nu\right) a_1\right] \\ &b_2 = (b_0 + y_w^2 b_1)(\ln y_w)' - y_w^2 b_3 \end{aligned} \tag{4.3}$$

The factor to b_0' in the first equation becomes zero for $z = 0$, so the initial condition that provides finiteness and continuity in the solution is derived by equating to zero the right part of this equation for the same section; then, as $y_w'(0) = 0$, we find $a_0'(0)$ from (4.2) to get

$$b_0(0) = -\frac{1+\nu}{2(3+\nu)}, \quad b_1(0) = \frac{1}{2}, \quad b_2(0) = -b_3(0) = -\frac{\varkappa_e+1}{2(3+\nu)} a_0'(0)$$

$$a_1(0) = 0, \quad a_0'(0) = \left(\frac{1+\nu}{\varkappa_e+1}\right)^{1/2} \tag{4.4}$$

Integration of (4.2) and of the first equation in (4.3) is carried from the minimum section $z = 0$ independently towards the negative and positive directions of z; one proceeds from the initial section using expansions in terms of z for a_0 and b_0. If we take z sufficiently small, we need consider only two terms, and the $b_0'(0)$ that needs to be added to (4.4) in that case is

$$b_0'(0) = -\frac{3+2\nu}{2(3+\nu)} a_0'(0)$$

We give below the expressions used in the calculation, which give the values of the derivatives for the other coefficients for $z = 0$:

$$a_1'(0) = 1, \quad b_1'(0) = a_0'(0), \quad b_2'(0) = -\frac{3+2(1+\nu)(\varkappa_e+3)}{3(3+\nu)}$$

$$b_3'(0) = \frac{2[3+(1+\nu)(\varkappa_e+3)]}{3(3+\nu)}, \quad a_0''(0) = \frac{3-2\varkappa_e}{3} a_0'^2(0)$$

$$a_0'''(0) = \frac{9-12\varkappa_e+\varkappa_e^2}{6} a_0'^3(0) - \frac{3}{2} a_0'(0) \tag{4.5}$$

The following equations are obtained from (3.4) and (3.5) after converting to the variables z and w, replacing ε_i by $\varepsilon_i^\circ = \varepsilon_i\sqrt{\varepsilon}$, substituting from (4.1) for the equilibrium distributions of the parameters, using conditions (1.2), and discarding terms of orders ε^2, $\varepsilon\varepsilon_i^\circ$ and above:

$$\Delta\rho' = \varkappa_e p_{e0}(1 - M_0^2)^{-1}\Gamma_1, \quad \Delta p' = \varkappa_e p_{e0} M_0^2 \Delta\rho' + \Gamma_2$$

$$\Delta u' = -a_0\Delta\rho' + \Delta u(\ln a_0)', \quad \Delta\rho_s = \Delta\rho + \Gamma_3$$

$$\Delta w = y\Delta u(\ln y_w)', \quad \Delta T = \frac{\Delta p}{\rho_{e0}} - \frac{p_{e0}}{\rho_{e0}}\Delta\rho$$

$$\Delta h = \frac{\varkappa p_{e0}}{(\varkappa-1)\rho_{e0}}\left(\frac{\Delta p}{p_{e0}} - \Delta\rho\right), \quad \Delta u_s = \Delta u - \varepsilon_1^\circ a_0 a_0' \tag{4.6}$$

$$\Delta w_s = \Delta w - \varepsilon_1^\circ a_0(a_0'y_w' + a_0 y_w'')\, y / y_w$$

$$\Delta T_s = \Delta T - \varepsilon_2^\circ \frac{\varkappa_e - 1}{\varkappa_e m}\delta a_0^2 a_0'$$

Here and subsequently $M_0^2 = a_0^2\rho_{\Sigma e0}/\varkappa_e p_{e0}$ is the square of the Mach number for the equilibrium one-dimensional flow; the subscript zero is attached to parameters whose calculation is performed from (3.3) with omission of v_e and with u_e replaced by a_0, which corresponds to equilibrium values for the corresponding quantities as obtained in the one-dimensional approximation. The functions Γ_i in (4.6) are

$$\Gamma_1 = \left[\frac{\Delta p}{p_{e0}} - m\Delta\rho - (1-m)\Delta\rho_s - 2\frac{\Delta u}{a_0} + \frac{\delta(\varkappa_e-1)^2(1-m)}{m\varkappa_e}(\Delta\rho - \Delta\rho_s)\right]\rho_{\Sigma e0}a_0 a_0' + \Phi_1 + \Phi_2$$

$$\Gamma_2 = \Phi_1 - \left[2\frac{\Delta u}{a_0} + m\Delta\rho + (1-m)\Delta\rho_s\right]\rho_{\Sigma e0}a_0 a_0'$$

$$\Gamma_3 = \varepsilon_1^\circ\left[(1+\nu)\left(a_1' + \frac{a_1^2}{a_0}\right) + a_0'' + (1-M_0^2)\frac{a_0'^2}{a_0}\right]$$

$$\Phi_1 = \varepsilon_1^\circ(1-m)(2a_0'^2 + a_0 a_0'')\rho_{\Sigma e0}a_0$$

$$\Phi_2 = (1-m)(1-\varkappa_e)\left\{\varepsilon_1^\circ a_0'^2 + \delta\frac{1-\varkappa_e}{m\varkappa_e}\left[\varepsilon_2^\circ\delta a_0 a_0'' + (\varepsilon_1^\circ + 2\varepsilon_2^\circ\delta)a_0'^2\right]\right\}\rho_{\Sigma e0}a_0$$

If $\Gamma_1(0) \neq 0$, the right part of the first equation in (4.6) becomes infinite in the section $z = 0$, where $M_0 = 1$; therefore, as in the case of (4.3), we have to put $\Gamma_1(0) = 0$ in order to provide a bounded and continuous solution, which gives a boundary condition for the integration of (4.6). The other boundary conditions are formulated in accordance with (1.3) and (1.4) as linear relationships between the Δu, Δp, ... for $z = -\infty$; for instance, it follows from (1.3) that for $z = -\infty$ we have

$$\Delta u_s = \Delta u, \quad \Delta w_s = \Delta w, \quad \Delta T_s = \Delta T$$

In this approximation, these equations are met automatically by virtue of the last three equations of (4.6), since in these cases we lose the added terms standing on the right in these equations for $z = -\infty$. On the other hand, the relationships following from (1.4) are not obeyed identically, and they give the three lacking boundary conditions at $z = -\infty$. The solution to the resulting boundary-value problem is difficult to obtain, since there is one condition at $z = 0$ and three at $z = -\infty$, but it can be carried through, as is done below, by substituting a Cauchy problem, which gives the lacking boundary conditions at $z = 0$, for instance, by putting here $\Delta\rho$, $\Delta\rho_s$, and Δ_u as zero. As a result we have

$$\Delta\rho = \Delta\rho_s = \Delta u = 0, \quad \Delta p = -(\Phi_1 + \Phi_2)/\varkappa_e a_0' \quad (z = 0) \tag{4.7}$$

We move away from the section $z = 0$, as in the case of (4.2) and (4.3), by using series expansions; the following equation gives the derivative $\Delta\rho'$ at $z = 0$, which is necessary in accordance with (4.6) and (4.7):

$$\Delta\rho' = \Gamma_4/2\,(1 + \varkappa_e)a_0'$$

where all the quantities are taken at $z = 0$, while

$$\Gamma_4 = \left\{(1-m)\left[1 + \delta\frac{(\varkappa_e - 1)^2}{m\varkappa_e}\right]\Phi_3 - \varkappa_e\Phi_1\right\}a_0' - \Phi_1' - \Phi_2' - \varkappa_e a_0'' - \varkappa_e^2 a_0'^2$$

$$\Phi_1' = \varepsilon_1^\circ(1-m)(5a_0'a_0'' + a_0''')$$

$$\Phi_2' = 2\varepsilon_1^\circ(1-m)(1-\varkappa_e)a_0'a_0'' + \delta\frac{(1-m)(\varkappa_e-1)^2}{m\varkappa_e}[\varepsilon_2^\circ a_0'''\delta + (2\varepsilon_1^\circ + 5\varepsilon_2^\circ\delta)a_0'a_0'']$$

$$\Phi_3 = \varepsilon_1^\circ[(1+\nu)(a_1' + a_1^2) + a_0'']$$

The values of the derivatives a_0'' and a_0''' at $z = 0$ are calculated via (4.5); it is readily seen that all the quantities different from $\Delta\rho$, Δp, Δu, and $\Delta\rho_S$ in the differential equations of (4.6) and the initial conditions of (4.7) are not dependent on y. Then the other relationships of (4.6) imply that only Δw and Δw_S are dependent on y, while the other terms are functions of z alone.

Of course, the values of the parameters at $z = -\infty$ obtained from solution of the Cauchy problem do not satisfy the conditions of (1.4), and these in that case are used to calculate the corrected values for m, the corrections being of order ε_1° and ε_2°, together with the critical velocity q_* and the density ρ_* of the mixture. The latter are found from the first two equations of (1.4), as written as

$$\frac{\varkappa_e p}{(\varkappa_e - 1)\rho_\Sigma} + \frac{q^2}{2} = \frac{(\varkappa_e + 1)q_*^2}{2(\varkappa_e - 1)}; \qquad \frac{p}{(\rho_\Sigma)^{\varkappa_e}} = \frac{q_*^2}{\varkappa_e\rho_*^{\varkappa_e - 1}}$$

We now show how one can take into account the correction of order $0(\varepsilon_i)$ in calculating the parameters in the wall layer with hardly any change in the relationships previously derived; in accordance with (3.7), the pressure in the wall layer in this zeroth approximation is constant and equal to the equilibrium value at the wall; incorporation of terms of higher order of smallness gives

$$\partial p / \partial n = \rho U^2 K$$

where K is the wall curvature, as in (2.3).

If the nozzle has a sufficiently smooth contour, the curvature at any point is of the same order as that in the minimal section, or even less, so $K \sim \varepsilon$; therefore, taking into account the thickness of the wall layer ($n_d \sim \varepsilon_1$), we get that the difference $(p_w - p_d)$ is zero also in the next approximation. As a consequence, to incorporate the first-order terms on the right in (3.7), (3.8), and (3.9) we have to substitute for the quantities with subscript we the corresponding parameters on the separation line, which are found incorporating Δp, $\Delta\rho$, ... and the displacement effect of the wall layer; the last, and also the Δp, $\Delta\rho$, ..., can here be calculated in the one-dimensional approximation, neglecting quantities of higher order. As one has to find $(\psi - \psi_w)$ in order to determine U and ρ with accuracy ε_1 inclusive by employing an accuracy ε_1^2, the y_w in (3.10) has, in accordance with (2.2), to be replaced by $y = y_w - n\cos\sigma$; then y_d, and hence also n_d, are determined by combined integration of (2.1) and the fifth and sixth equations in (1.1), in which one substitutes u and v calculated for the core of the flow with accuracies ε_1 and ε_2 inclusive. Finally, $V \sim \varepsilon_1$ is found by integration with respect to n in the last equation in (2.3), which can be given the following form in conjunction with the other equations in this system:

$$\left(\frac{\partial V}{\partial \psi}\right)_\tau = \frac{M^2-1}{cy_w{}^\nu \rho^2 U^2}\frac{dp_{we}}{d\tau} + \frac{V}{U}\left(\frac{\partial U}{\partial \psi}\right)_\tau \quad \left(M^2 = \frac{\rho U^2}{\varkappa p_{we}}\right)$$

All the quantities in this equation, except V, are taken in the zeroth approximation; V is zero at the wall, i.e., for $(\psi = \psi_w)$, in accordance with (2.6).

5. The approach developed here has been applied to calculate the flow of a two-phase mixture in an axially symmetrical hyperbolic nozzle, whose shape was specified as $y_w = (1 + 0.2\,x^2)^{1/2}$, which corresponds to $\varepsilon = 0.2$. The shape was symmetrical about the y axis, and for $|x| \to \infty$ had as asymptotes straight lines with $|y_w'| = \sqrt{5}$; the constants in the equations were taken as $\varkappa = 1.2$, $\delta = 1.0$, $m = 0.25$, and $\varphi^f = \varphi^q = 10$, which gives $\varepsilon_1 = \varepsilon_2 = 0.1$ and $\varkappa_e = 1.125$.

The calculations were performed with an M-20 computer, incorporating the terms of zero and first orders in ε_1, ε_2, and ε for the core and the wall layer; the differential equations of (4.2), (4.3), and (4.6) were integrated from the section $z = 0$ by the Runge–Kutta method; in this case, the wall layer actually appears at $x = -\infty$, but the calculation actually began at $x_0 = -1.0$, where it was assumed that $n_d = 0$; it was found from analogous calculations with $x_0 = -0.5$ and -2.0 that this choice provides an accuracy quite sufficient for graphical representation of the results.

Figures 2-5 show some of the results; in Fig. 2 the scale on the y axis is twice that along the x axis, while the x axis itself has been brought into coincidence with the line $y = 0.5$; this shows the nozzle contour, the interface (broken line), and the lines of constant Mach number (the corresponding values are given as figures near the curves). The Mach number $M = q/a$ was calculated here from the speed of sound in the gas $a = (\varkappa p/\rho)^{1/2}$, which defines [4-14, 20] the type of system in (1.1). In this case, the region occupied by the particles has a gas speed (and especially a particle speed) that remains subsonic very far into the expanding part of the nozzle; on the other hand, the value unity is attained near the minimal section by q/a_Σ, where $a_\Sigma = (\varkappa_e p/\rho_\Sigma)^{1/2}$ is the equilibrium velocity of a sound in the mixture (this meaning for a_Σ applies only for those points in the flow at which there is no thermal or velocity lag by particles).

The solid, broken, and dot-and-dash lines in Fig. 3 represent the variations with x for the nonequilibrium terms $(\Delta\varphi)$, the pressure, and the axial components of the velocities for the gas and particles respectively; the maximum difference $u - u_s = \Delta u - \Delta u_s$ occurs near the minimum section of the nozzle, which corresponds with known results from one-dimensional calculations. The above analysis shows that in this approximation the incorporation of the particle lag effects is also essentially one-dimensional. The values of Δu and Δu_s and also the velocities of gas and particles at $x \to -\infty$ tend to zero, while $\Delta p(-\infty) = -0.13$.

Figure 4 gives a more complete representation than that of Fig. 2 for the nonuniformity of the flow in the wall layer; it shows U varying with n for various sections of the nozzle (the numbers on the curves are the values of x_w). In considering Fig. 4 it should be borne in mind that U has been referred to the critical velocity of the equilibrium two-phase flow. The broken line in Fig. 4 corresponds to the interface. Below this, the lines giving the distribution of U in n for τ = constant are almost parallel to the horizontal axis, which is due to the small nonuniformity of the parameters at the core of the flow. This is clear also from Fig. 5, which shows the variation in M along the symmetry axis (solid line), along the interface (dotted line), and at the wall (dot-and-dash line). Other results [10, 11, 14, 21] confirm this for the supersonic part of the flow in a Laval nozzle, and also the marked nonuniformity in the parameters near the wall when there is a high particle content; this nonuniformity decreases as m increases, i.e., as the flow of particles decreases.

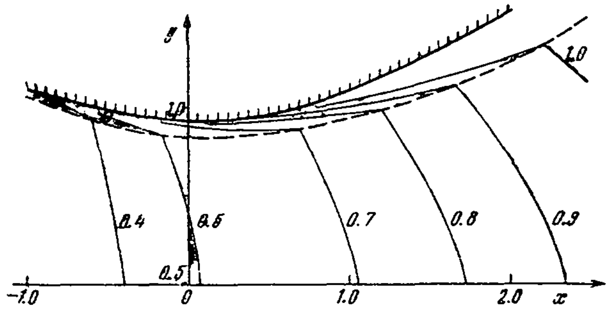

Fig. 2

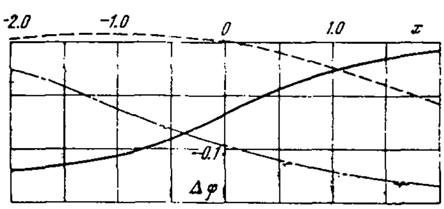

Fig. 3

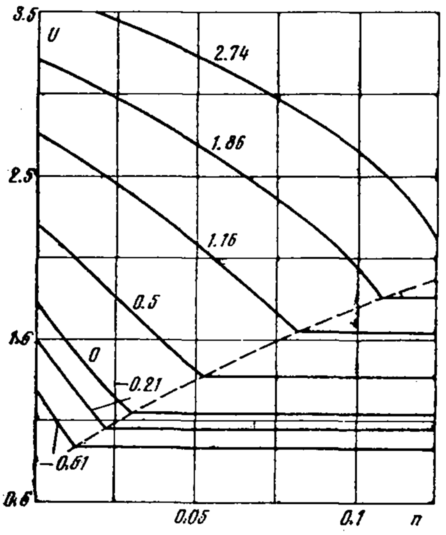

Fig. 4

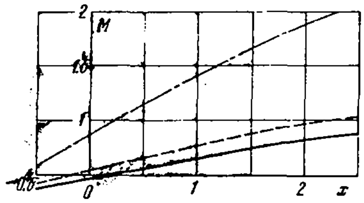

Fig. 5

As a two-phase flow gives rise to a comparatively thin wall layer, one naturally has to examine the roles of the viscosity and the thermal conductivity of the gas, which have been taken into account in the above model only for the phase interaction; in fact, these dissipative effects are important only near the wall, which is the reason for the formation of a viscous boundary layer. The above analysis for the flow in the wall layer of pure gas applies only when this layer is substantially thicker than the viscous boundary layer formed at the wall. The thickness of the latter is determined by the Reynolds number, which is a dimensionless parameter independent of the dimensionless parameters characterizing the phase interaction, so both situations can arise.

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